ECE-6554 Planar Bi-Rotor Helicopter Project Step 2

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Plant Physics:

$$m\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = -d\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} + \begin{bmatrix}-\sin(\theta) & -\sin(\theta)\\ \cos(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} - m\begin{bmatrix}0\\ g\end{bmatrix}$$

$$J\ddot{\theta} = r(f_x - f_y)$$

Parameters:

$$\begin{aligned} m &= 6\\ d &= 0.1\\ g &= 9.8\\ r &= 0.25\\ J &= 0.1425 \end{aligned}$$

Transformed Coordinates for Differential Flatness:

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}x\\ y\end{bmatrix} + \lambda\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix}x - \lambda\sin(\theta)\\ y + \lambda\cos(\theta)\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}'\\ \dot{y}'\end{bmatrix} = \begin{bmatrix}\dot{x} - \lambda\dot{\theta}\cos(\theta)\\ \dot{y} - \lambda\dot{\theta}\sin(\theta)\end{bmatrix}$$

$$\begin{bmatrix}\ddot{x}'\\ \ddot{y}'\end{bmatrix} = \begin{bmatrix}\ddot{x} - \lambda\left(\cos(\theta)\ddot{\theta} - \sin(\theta)\dot{\theta}^2\right)\\ \ddot{y} - \lambda\left(\sin(\theta)\ddot{\theta} + \cos(\theta)\dot{\theta}^2\right)\end{bmatrix} \quad \text{Plant :} \begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = -\frac{d}{m}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} + \frac{1}{m}\begin{bmatrix}-\sin(\theta) & -\sin(\theta)\\ \cos(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} - \begin{bmatrix}0\\ g\end{bmatrix}$$

$$\begin{bmatrix}\ddot{x}'\\ \ddot{y}'\end{bmatrix} = \left[-\frac{d}{m}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} + \frac{1}{m}\begin{bmatrix}-\sin(\theta) & -\sin(\theta)\\ \cos(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} - \begin{bmatrix}0\\ g\end{bmatrix}\right] - \lambda\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}\dot{\theta}^2 - \lambda\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\ddot{\theta}; \quad \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\dot{x}'\\ \dot{y}'\end{bmatrix} + \lambda\dot{\theta}\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

$$\begin{bmatrix}\ddot{x}'\\ \ddot{y}'\end{bmatrix} = -\frac{d}{m}\begin{bmatrix}\dot{x}'\\ \dot{y}'\end{bmatrix} - \frac{d}{m}\lambda\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\dot{\theta} + \frac{1}{m}\begin{bmatrix}-\sin(\theta) & -\sin(\theta)\\ \cos(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} - \begin{bmatrix}0\\ g\end{bmatrix} - \lambda\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}\dot{\theta}^2 - \lambda\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\ddot{\theta}$$

$$\ddot{\theta} = \frac{r}{J}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

$$\begin{bmatrix}\ddot{x}'\\ \ddot{y}'\end{bmatrix} = -\frac{d}{m}\begin{bmatrix}\dot{x}'\\ \dot{y}'\end{bmatrix} - \frac{d}{m}\lambda\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\dot{\theta} - \lambda\begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}\dot{\theta}^2 + \frac{1}{m}\begin{bmatrix}-\sin(\theta) & -\sin(\theta)\\ \cos(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} + \frac{\lambda r}{J}\begin{bmatrix}-\cos(\theta) & \cos(\theta)\\ -\sin(\theta) & \sin(\theta)\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} - \begin{bmatrix}0\\ g\end{bmatrix}$$

$$\begin{bmatrix} \ddot{x}' \\ \ddot{y}' \end{bmatrix} = -\frac{d}{m}\begin{bmatrix} \dot{x}' \\ \dot{y}' \end{bmatrix} - \frac{d}{m}\lambda\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}\dot{\theta} - \lambda\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}\dot{\theta}^2 + \begin{bmatrix} -\frac{1}{m}\sin(\theta) - \frac{\lambda r}{J}\cos(\theta) & -\frac{1}{m}\sin(\theta) + \frac{\lambda r}{J}\cos(\theta) \\ \frac{1}{m}\cos(\theta) - \frac{\lambda r}{J}\sin(\theta) & \frac{1}{m}\cos(\theta) + \frac{\lambda r}{J}\sin(\theta) \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\begin{bmatrix} f_{\mathrm{LR}} \\ f_{\mathrm{UD}} \end{bmatrix} := \begin{bmatrix} -\frac{1}{m}\sin(\theta) - \frac{\lambda r}{J}\cos(\theta) & -\frac{1}{m}\sin(\theta) + \frac{\lambda r}{J}\cos(\theta) \\ \frac{1}{m}\cos(\theta) - \frac{\lambda r}{J}\sin(\theta) & \frac{1}{m}\cos(\theta) + \frac{\lambda r}{J}\sin(\theta) \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} := M\begin{bmatrix} f_1 \\ f_2 \end{bmatrix};$$

$$\ddot{\theta} = \frac{r}{J}\begin{bmatrix} 1 & -1 \end{bmatrix} M^{-1}\begin{bmatrix} f_{\mathrm{LR}} \\ f_{\mathrm{UD}} \end{bmatrix} = \frac{1}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}}\begin{bmatrix} -\cos(\theta) & -\sin(\theta) \end{bmatrix}\begin{bmatrix} f_{\mathrm{LR}} \\ f_{\mathrm{UD}} \end{bmatrix}$$

State-Space form:

Let:

$$x_1 = x'$$
$$x_2 = \dot{x}_1'$$
$$x_3 = y'$$
$$x_4 = \dot{y}'$$
$$x_5 = \theta'$$
$$x_6 = \dot{\theta}'$$

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x' \\ \dot{x}' \\ y' \\ \dot{y}' \\ \theta' \\ \dot{\theta}' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \frac{-\cos(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} & \frac{-\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} \end{bmatrix}\begin{bmatrix} f_{\mathrm{LR}} \\ f_{\mathrm{UD}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{d}{m}\lambda\cos(\theta)\dot{\theta} + \lambda\sin(\theta)\dot{\theta}^2 \\ 0 \\ -\frac{d}{m}\lambda\sin(\theta)\dot{\theta} - \lambda\cos(\theta)\dot{\theta}^2 - g \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{x} = g(x, f); f = \begin{bmatrix} f_{\mathrm{LR}} \\ f_{\mathrm{UD}} \end{bmatrix}$$

Fixed point calculation:

At Equilibrium $x = x_e, f = u_e, \dot{x} = 0$:

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{d}{m}x_2 - f_{\text{LR}} - \frac{d}{m}\lambda\cos(\theta)\dot{\theta} + \lambda\sin(\theta)\dot{\theta}^2 \\ x_4 \\ -\frac{d}{m}x_4 + f_{\text{UD}} - \frac{d}{m}\lambda\cos(\theta)\dot{\theta} + \lambda\sin(\theta)\dot{\theta}^2 - g \\ x_6 \\ \frac{-\cos(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} f_{\text{LR}} + \frac{-\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} f_{\text{UD}} \end{bmatrix} = 0$$

$$\Rightarrow x_2 = x_4 = x_6 = 0$$

$$\text{Given} : x_5 = \theta = \dot{\theta} = 0$$

$$\begin{bmatrix} 0 \\ -f_{\text{LR}} \\ 0 \\ f_{\text{UD}} - g \\ 0 \\ \frac{-1}{\frac{\lambda r}{J}} f_{\text{LR}} \end{bmatrix} = 0$$

$$\Rightarrow f_{\text{LR}} = 0$$

$$\Rightarrow f_{\text{UD}} = g$$

$$\text{Let} : x_1 = x_3 = 0$$

Therefore: $x_e = 0, u_e = \begin{bmatrix} 0 \\ g \end{bmatrix}$

Linearization

$$x = x_e + \delta x,\ f = u_e + \delta u$$

$$\dot{x} = g(x, f)$$

$$\dot{x_e} + \dot{\delta x} = g(x_e + \delta x, u_e + \delta u) = g(x_e, u_e) + D_x g|_{(x_e, u_e)}\delta x + D_f g|_{(e_e, u_e)}\delta u$$

$$\dot{\delta x} = D_x g|_{(x_e, u_e)}\delta x + D_f g|_{(e_e, u_e)}\delta u = A\delta x + B\delta u$$

$$\frac{\partial g(x,f)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \frac{\partial g(x,f)}{\partial f} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \frac{-\cos(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} & \frac{-\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} \end{bmatrix}$$

Linearization at $x = x_e = 0, x = x_e + \delta x \Rightarrow \delta x = x$

$$D_x g = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D_f g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -\frac{J}{r\lambda} & 0 \end{bmatrix}$$

Linearization of input. $f = u_e \Rightarrow f_{\mathrm{LR}} = 0, f_{\mathrm{UD}} = g$

Let: $u = \delta u = f - u_e \Rightarrow u_1 = f_{\mathrm{LR}}, u_2 = f_{\mathrm{UD}} - g$

$$\dot{\delta x} = D_x g|_{(x_e, u_e)} \delta x + D_f g|_{(e_e, u_e)} \delta u = A\delta x + B\delta u$$
$$\delta x = x, \delta u = u$$
$$\Rightarrow \dot{x} = Ax + Bu$$

$$\text{Let}: \lambda = \frac{J}{r} = 0.57$$

$$\begin{bmatrix} \dot{x}' \\ \dot{x}' \\ y' \\ \dot{y}' \\ \theta' \\ \dot{\theta}' \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

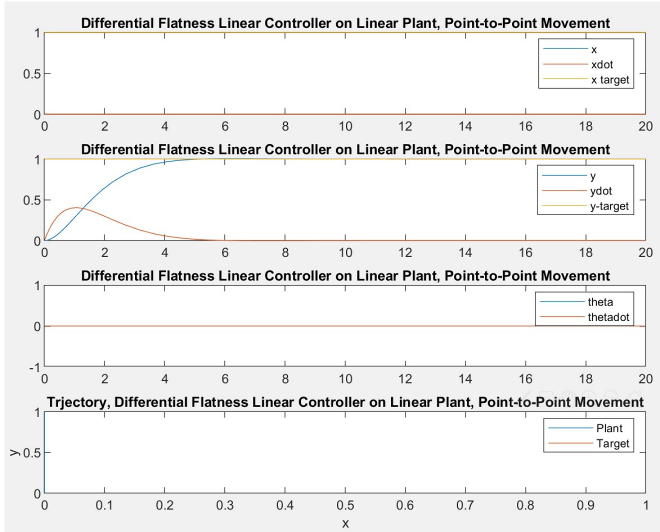
Gain Calculation from Cost: LQR and CARE

Like in step 1, because the non-linear system behaves linearly near $x_5 = 0$, our controller gains needs to be set so that it prefers $\theta = 0, \dot{\theta} = 0$ over the other state variables.

And again, since the thrusters cannot produce negative forces, and the thrusters cannot produce six-times more than countering gravity, a high cost of R would result in low u. Keeping in mind that $f \geq 0 \Rightarrow u \geq -\frac{mg}{2}$

$$-\frac{mg}{2} \leq u_1 \leq 5\frac{mg}{2}, -\frac{mg}{2} \leq u_2 \leq 5\frac{mg}{2}$$

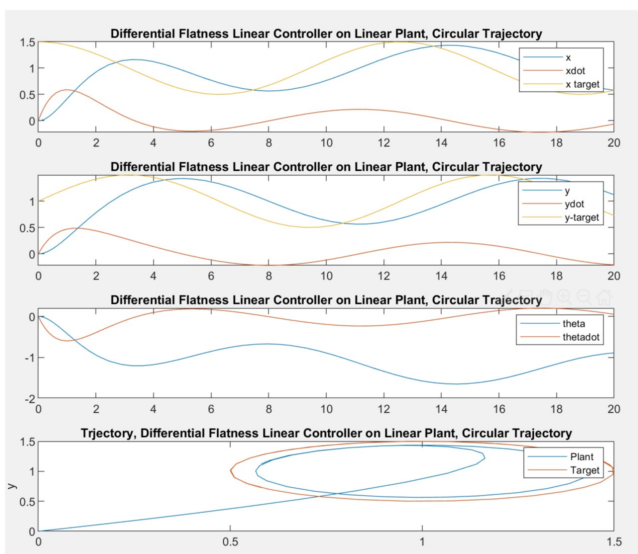
$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



However, after applying LQR to the system, the calculated gains do not have a gain for the x component. Hence, the x component does non change ever.

To fix this, the θ component of the transformed linearized system will be ignored. Although the transformed coordinates and the actual one share the same theta, it is not the goal of the transformed system.

$$\begin{bmatrix} \dot{x}' \\ \dot{x}' \\ y' \\ \dot{y}' \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



And thus, the x' and y' are stabilized, at the cost of theta.
This will be fixed by manipulation of the non-linearities.

Manipulating Differential Flatness for Non-linear Control

$$x_e = 0, x = x_e + \delta x = \delta x, u = \delta u, f = u_e + \delta u = u_e + u$$

$$\dot{x} = g(x, f)$$

$$\dot{x} = \dot{x_e} + \dot{\delta x} = \dot{x_e} + D_x g|_{(x_e,u_e)}\delta x + D_f g|_{(e_e,u_e)} u$$

$$\dot{x} = Ax + Bu + \dot{x_e}$$

$$\text{We want : } \dot{x} = Ax + Bu + \dot{x_e} = Ax + B(u + \alpha^T \phi)$$

$$\text{Therefore : } Bu + g(x_e, u_e) = B(u + \alpha^T \phi)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \frac{-\cos(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} & \frac{-\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} \end{bmatrix} \begin{bmatrix} f_{\text{LR}} \\ f_{\text{UD}} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{d}{m}\lambda\cos(\theta)\dot{\theta} + \lambda\sin(\theta)\dot{\theta}^2 \\ 0 \\ -\frac{d}{m}\lambda\sin(\theta)\dot{\theta} - \lambda\cos(\theta)\dot{\theta}^2 - g \\ 0 \\ 0 \end{bmatrix}$$

We Need:

$$\begin{bmatrix} 0 \\ f_{\text{LR}} - \frac{d}{m}\lambda\cos(\theta)\dot{\theta} + \lambda\sin(\theta)\dot{\theta}^2 \\ 0 \\ f_{\text{UD}} - \frac{d}{m}\lambda\sin(\theta)\dot{\theta} - \lambda\cos(\theta)\dot{\theta}^2 - g \\ 0 \\ \frac{-\cos(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} f_{\text{LR}} + \frac{-\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} f_{\text{UD}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \alpha^T \phi \right)$$

$$\begin{bmatrix} 0 \\ u_1 - \frac{d}{m}\lambda\cos(\theta)\dot{\theta} + \lambda\sin(\theta)\dot{\theta}^2 \\ 0 \\ u_2 - \frac{d}{m}\lambda\sin(\theta)\dot{\theta} - \lambda\cos(\theta)\dot{\theta}^2 \\ 0 \\ \frac{-\cos(\theta)u_1 - \sin(\theta)u_2 - g\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta) + \frac{\lambda r}{J}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \alpha^T \phi \right)$$

$$
\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \dfrac{-\cos(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta)+\frac{\lambda r}{J}} & \dfrac{-\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta)+\frac{\lambda r}{J}} u_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{d}{m}\lambda\cos(\theta)\dot{\theta}+\lambda\sin(\theta)\dot{\theta}^2 \\ 0 \\ -\frac{d}{m}\lambda\sin(\theta)\dot{\theta}-\lambda\cos(\theta)\dot{\theta}^2 \\ 0 \\ \dfrac{-g\sin(\theta)}{\frac{1}{m}\sin(\theta)\cos(\theta)+\frac{\lambda r}{J}} \end{bmatrix}
$$

$$
\text{We Want : } \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -\frac{d}{m}\lambda\cos(\theta)\dot{\theta}+\lambda\sin(\theta)\dot{\theta}^2 \\ -\frac{d}{m}\lambda\sin(\theta)\dot{\theta}-\lambda\cos(\theta)\dot{\theta}^2 \end{bmatrix} \right)
$$

Not possible unless we remove θ and $\dot{\theta}$

$$
\begin{bmatrix} \dot{x}_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{d}{m} & 1 & 0 & 0 \\ 0 & 0 & -\frac{d}{m} & -1 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta)\dot{\theta}^2 \\ \sin(\theta) \\ \cos(\theta)\dot{\theta}^2 \end{bmatrix} \right)
$$

$$
\alpha^T = \lambda \begin{bmatrix} -\frac{d}{m} & 1 & 0 & 0 \\ 0 & 0 & -\frac{d}{m} & -1 \end{bmatrix}, \phi(\theta, \dot{\theta}) = \begin{bmatrix} \cos(\theta)\dot{\theta} \\ \sin(\theta)\dot{\theta}^2 \\ \sin(\theta)\dot{\theta} \\ \cos(\theta)\dot{\theta}^2 \end{bmatrix}
$$

$$
u = K_x^T x + K_r^T r - \alpha^T \phi
$$

Gain Calculation from Cost: LQR and CARE for 2 Isolated systems

Theta is now isolated from x' and y'. Thus LQR and CARE are applied to the isolated systems.

$$
A' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{d}{m} \end{bmatrix}, B' = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, Q' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, R' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$A_\theta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_\theta = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, Q_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From LQR:

$$K' = \begin{bmatrix} 1 & 1.7155 & 0 & 0 \\ 0 & 0 & 1 & 1.7155 \end{bmatrix}, K_\theta = \begin{bmatrix} -1 & -1.7321 \\ 0 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} K' & K_\theta \end{bmatrix}$$

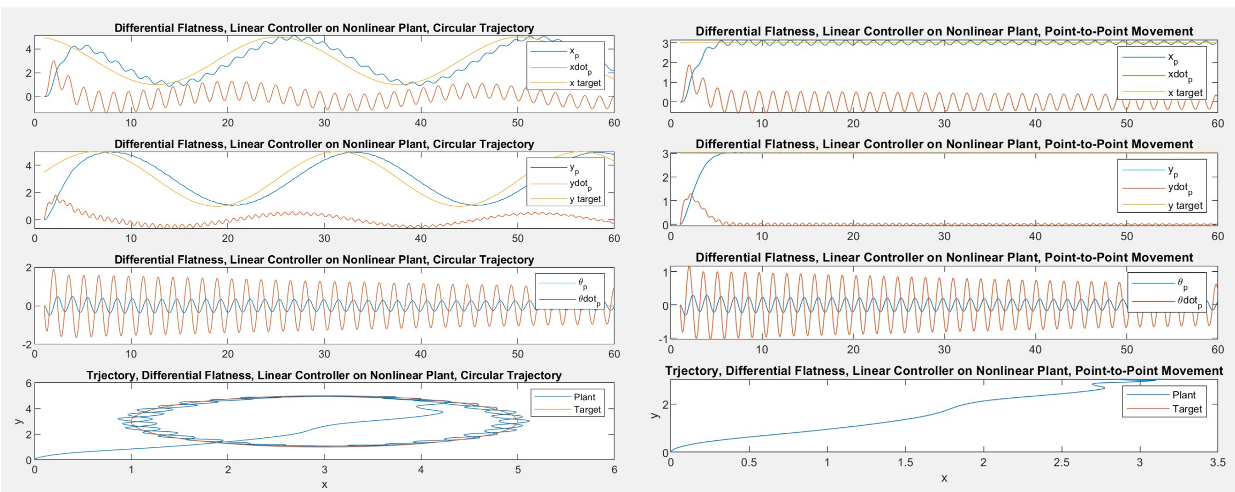
Since we desire $x \rightarrow r \Rightarrow K = K_x = K_r$

Similarly, P was calculated using CARE, used in the adaptive contoller.

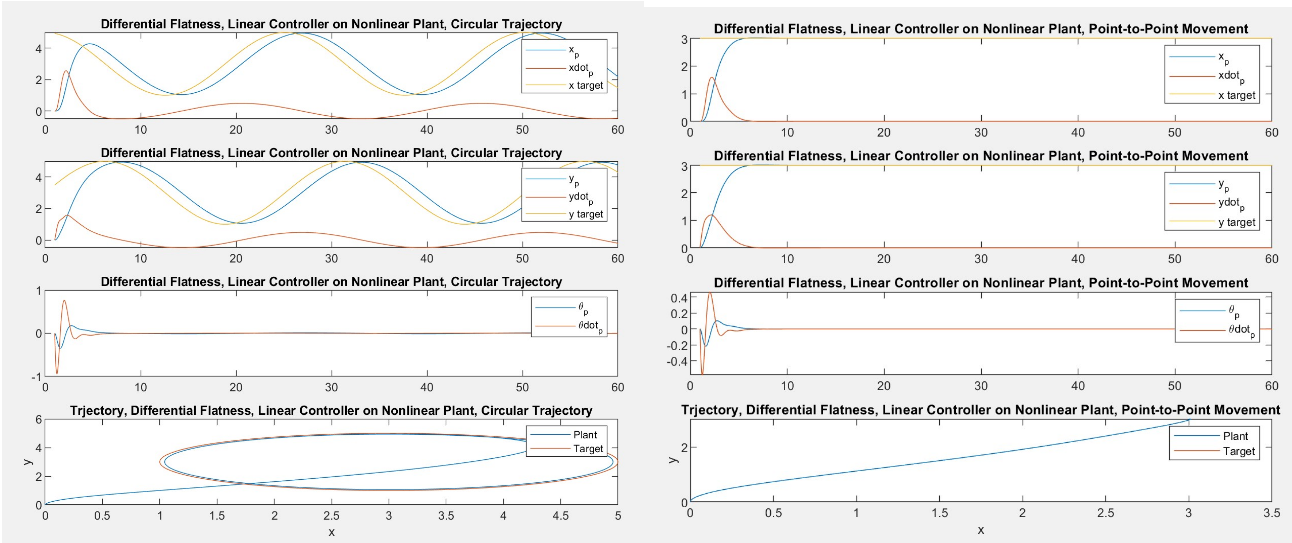
$$P' = \begin{bmatrix} 1.7321 & 1 & 0 & 0 \\ 1 & 1.7155 & 0 & 0 \\ 0 & 0 & 1.7321 & 1 \\ 0 & 0 & 1 & 1.7155 \end{bmatrix}, P_\theta = \begin{bmatrix} 1.7321 & 1 \\ 1 & 1.7321 \end{bmatrix}$$

$$P = \begin{bmatrix} P' & 0 \\ 0 & P_\theta \end{bmatrix}$$

The importance of K_θ is shown below, with a circular trajectory, and a point-to-point trajectory.



Controller without K_θ applied to non-linear plant.



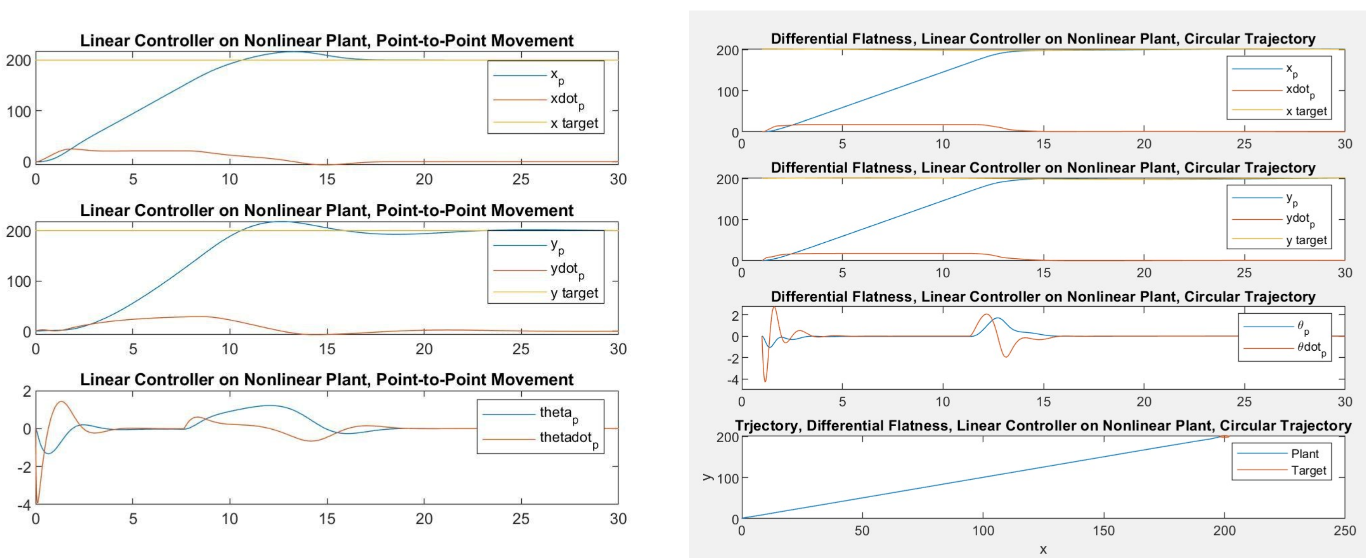
Controller with K_θ applied to non-linear plant.

Therefore, although u corrects the system for non-linear behavior, reducing theta still helps as it prevents wabbling.

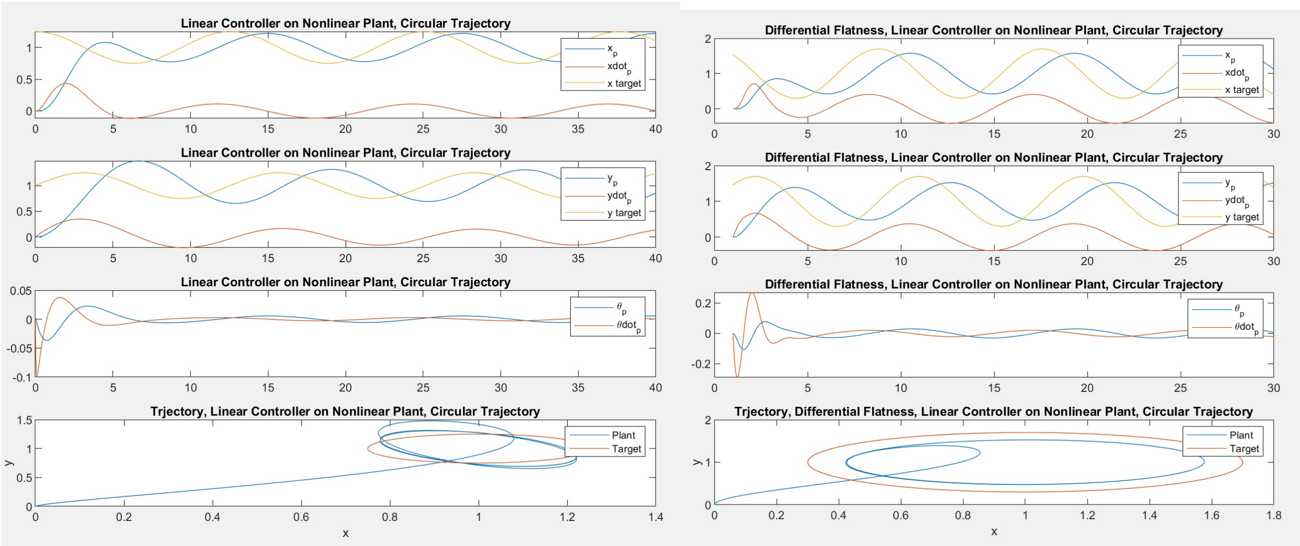
Although the linear plant cannot correct both x and θ this, a non-linear plant can, and thus the non-linearities can be manipulated

And just as in step 1, the reference command was capped to ± 30 of x and y.

Comparing Step 1 and Step 2: Linear Controller V.S. Differential Flatness with Non-linear Controller



Both reach the target in 15 seconds, but the differential flatness (right) along with a non-linear controller, allowed us to remove the non-linearities, so it performed better!



When given a circular tragectory, the path is not perfectly circular for Step 1 (left), but it is circular for Step 2 (right).

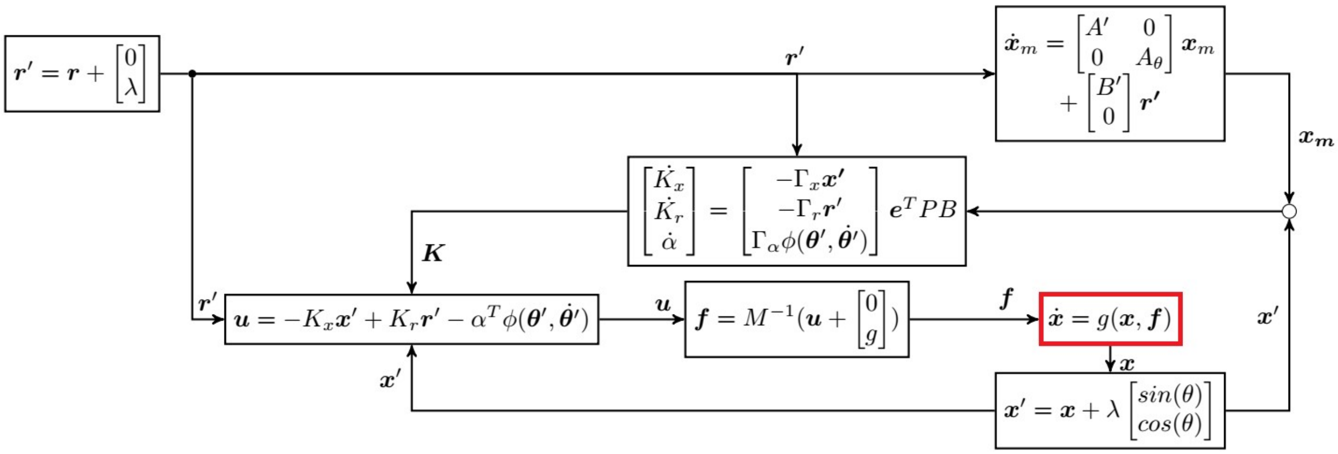
Hence dealing with the non-linearities allowed for much better control.

Adaptive Contoller

$$\phi\left(\theta,\dot{\theta}\right) = \begin{bmatrix} \cos(\theta)\dot{\theta} \\ \sin(\theta)\dot{\theta}^2 \\ \sin(\theta)\dot{\theta} \\ \cos(\theta)\dot{\theta}^2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, P' = \begin{bmatrix} \begin{bmatrix} 1.7321 & 1 & 0 & 0 \\ 1 & 1.7155 & 0 & 0 \\ 0 & 0 & 1.7321 & 1 \\ 0 & 0 & 1 & 1.7155 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1.7321 & 1 \\ 1 & 1.7321 \end{bmatrix} \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{1}{m}\sin(\theta) - \frac{\lambda r}{J}\cos(\theta) & -\frac{1}{m}\sin(\theta) + \frac{\lambda r}{J}\cos(\theta) \\ \frac{1}{m}\cos(\theta) - \frac{\lambda r}{J}\sin(\theta) & \frac{1}{m}\cos(\theta) + \frac{\lambda r}{J}\sin(\theta) \end{bmatrix},$$

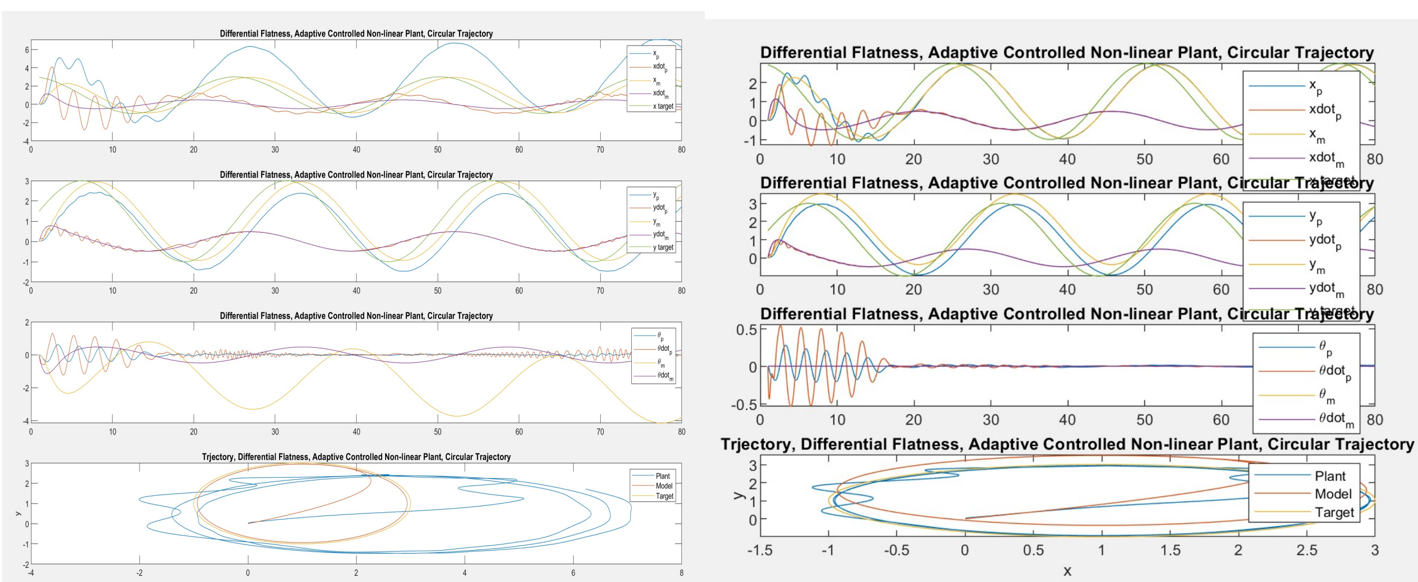
$$K_x^T(0) = K_r^T(0) = \left[\begin{bmatrix} 1 & 1.7155 & 0 & 0 \\ 0 & 0 & 1 & 1.7155 \end{bmatrix} \quad \begin{bmatrix} -1 & -1.7321 \\ 0 & 0 \end{bmatrix} \right], \alpha^T(0) = \lambda \begin{bmatrix} -\frac{d}{m} & 1 & 0 & 0 \\ 0 & 0 & -\frac{d}{m} & -1 \end{bmatrix}$$



The system with the adaptive contoller is shown above. A model (top-right) and adaptor (middle), were added to augment the non-linear controller (bottom 2 rows).

The plant is shown in red. Everything else is part of the contorller. Note that the controller uses the transformed coordinates of the state. Also note that $\theta = \theta'$ and it is just an entry of x and x'.

In the system above, the model does not implement B_θ because when it does, the result looks like the figure below on the left. Due to this, the B used in the model is not the same as the one in the adaptive gain update.

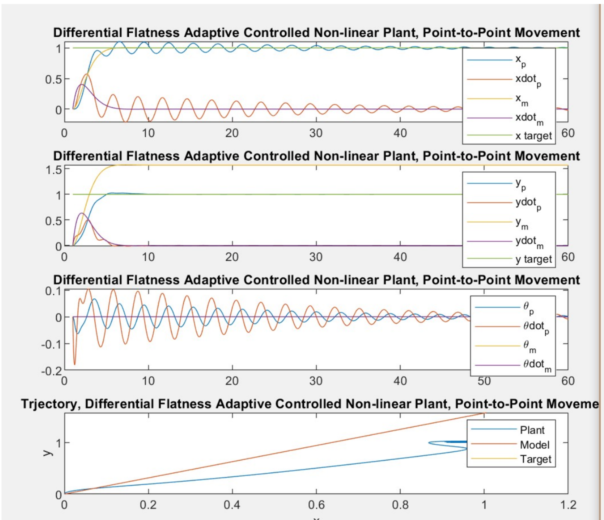


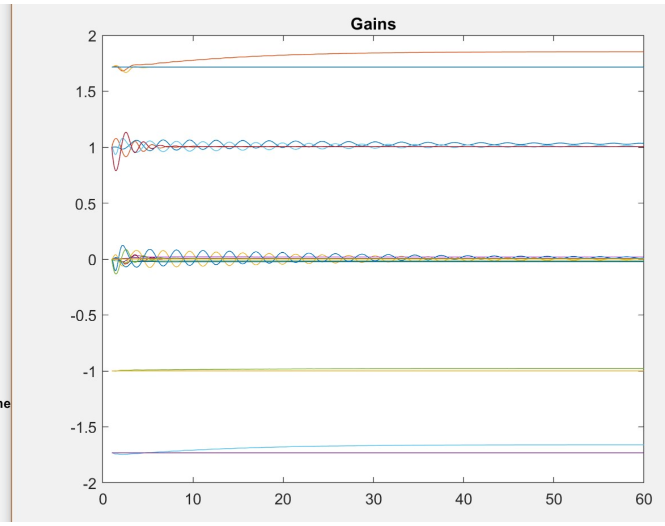
Left: Result if θ_m is affected by thrusters (by implementing B_θ) and the plant is far from the target. Right: Results if $\theta_m = 0$ always. The plant went to the target.

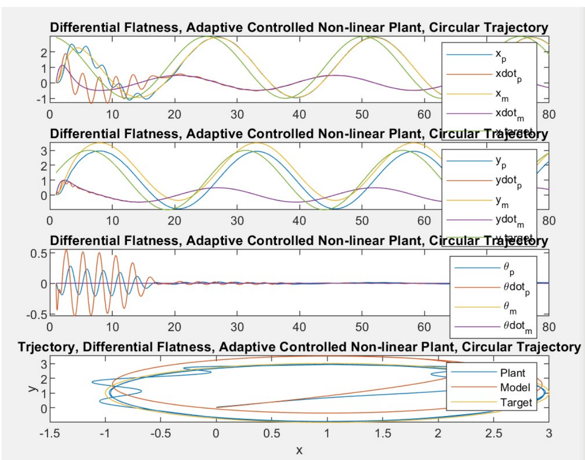
Even though in the linearization of the transformed coordinates, θ is dependent on u, better result are yeilded when $\theta_m = 0$, independent of u.

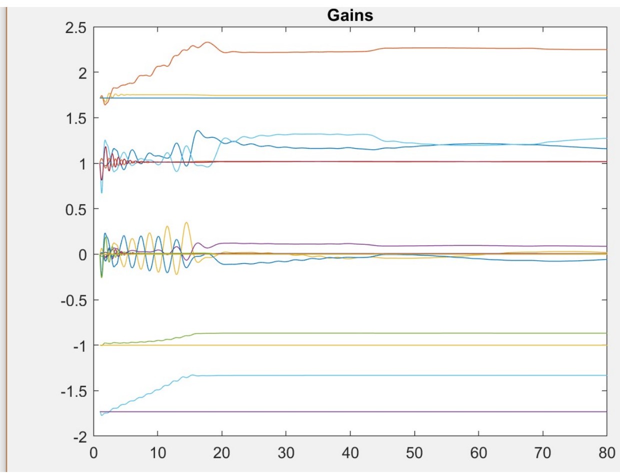
The result above shows that the model should follow the desired physics, even if it is not realistic with respect to the linearization, because the plant can deal with non-linearities, where the model cannot.

Adaptive Contoller Results



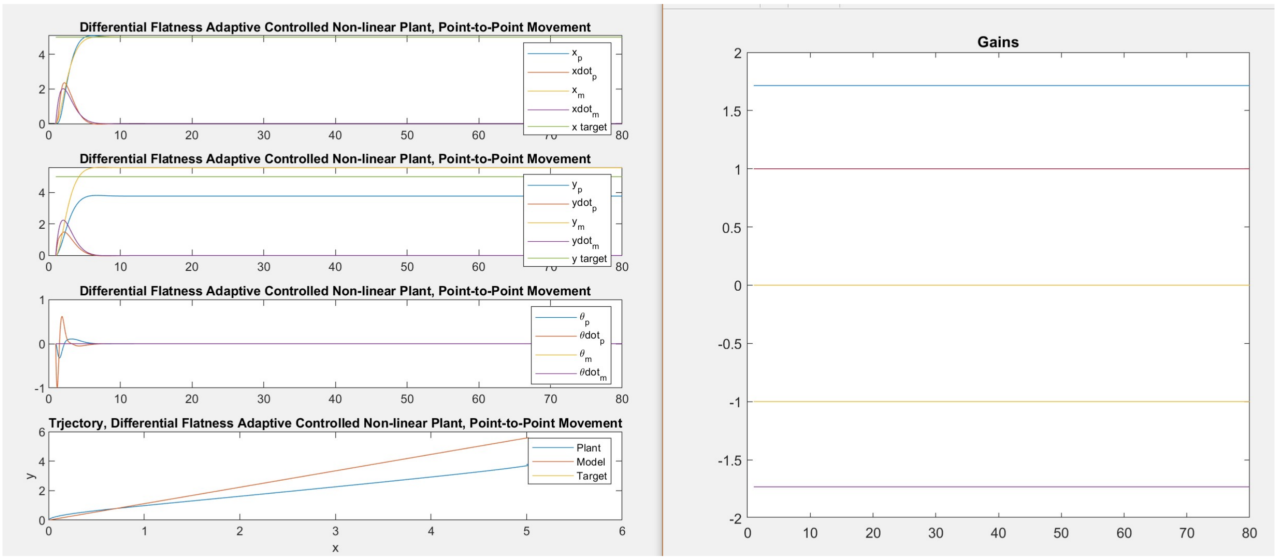


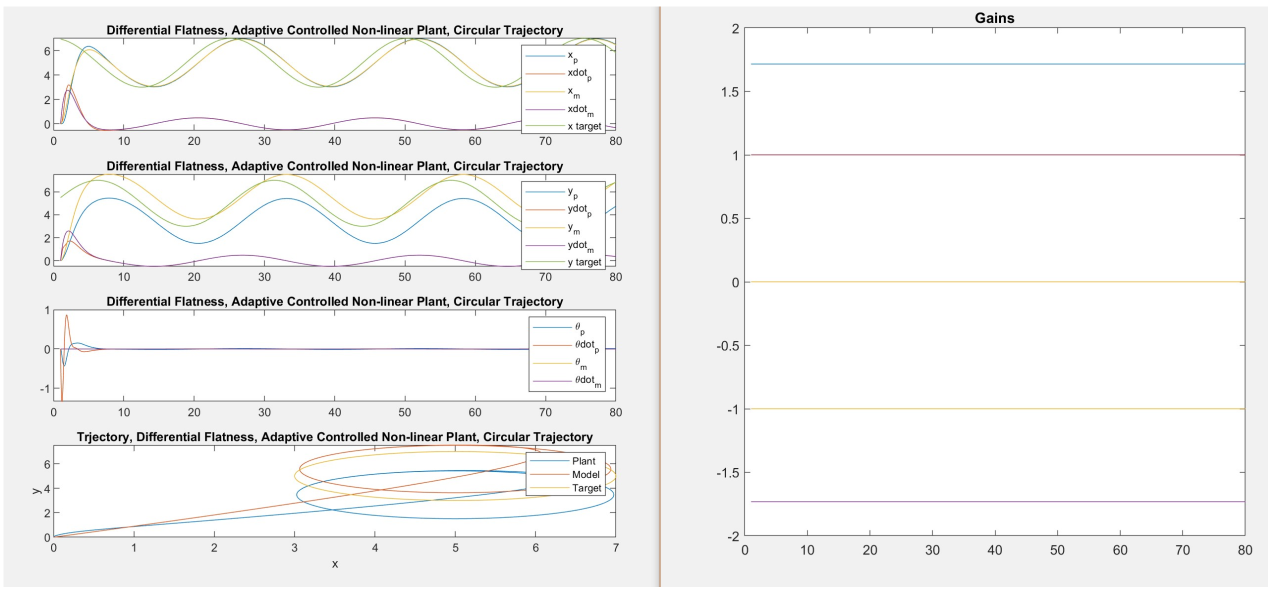




The adaptive and linear controller have both worked fairly well on the non-linear plant. So why use an adaptive contoller?

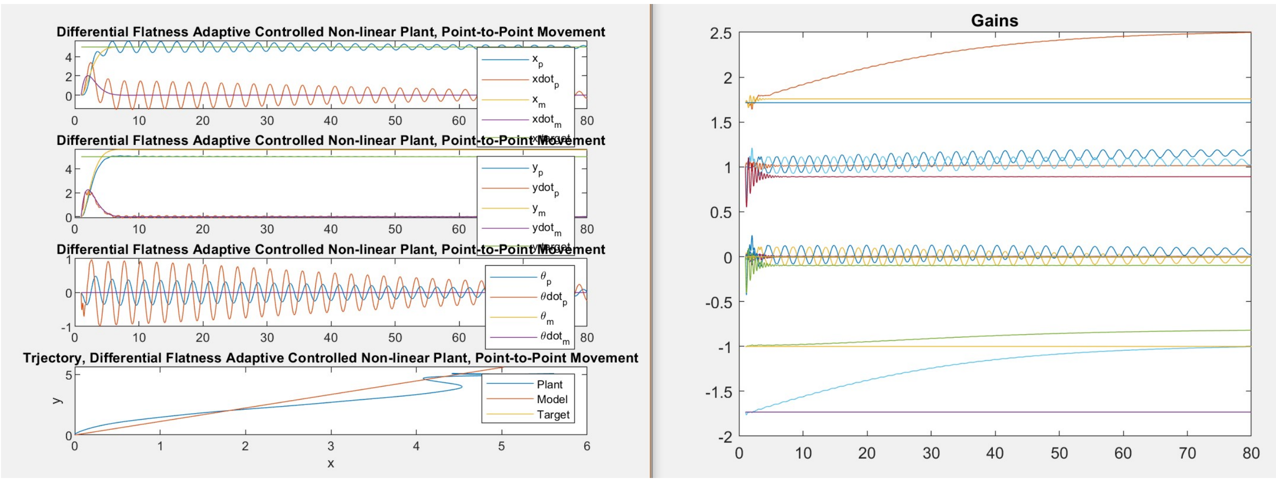
Physical parameters changed in plant by $\pm 20\%$ without adaptation

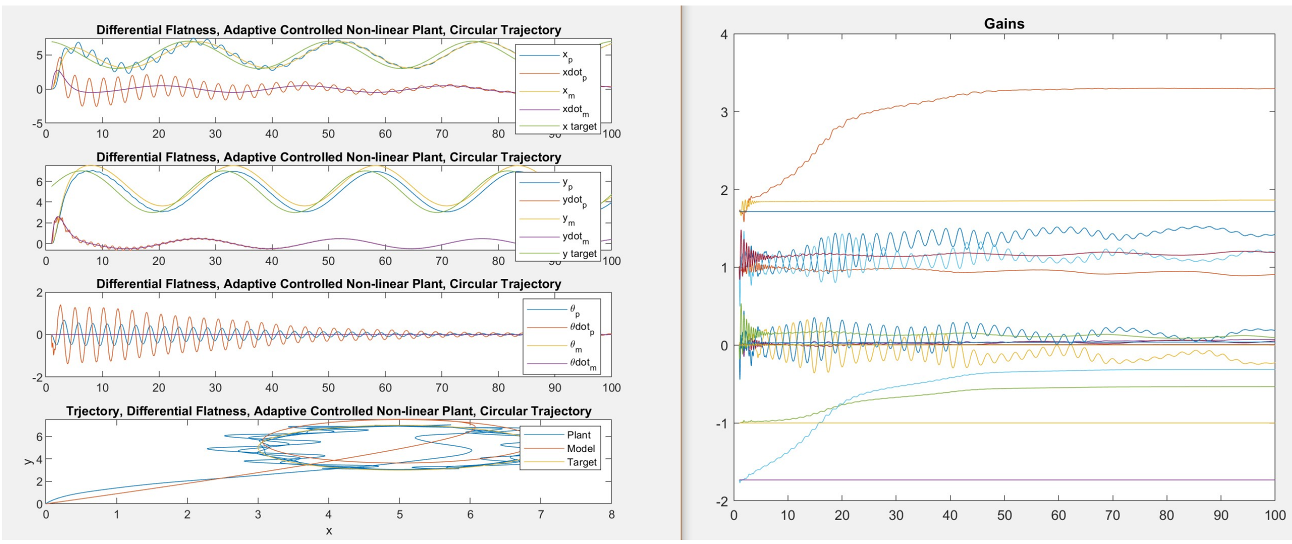




Here, the physical plant parameters have been altered by $\pm 20\%$ but the there is no adaptation. The controller fails to reach the target in both point-to-point and circular tragectories, Just like in step 1.

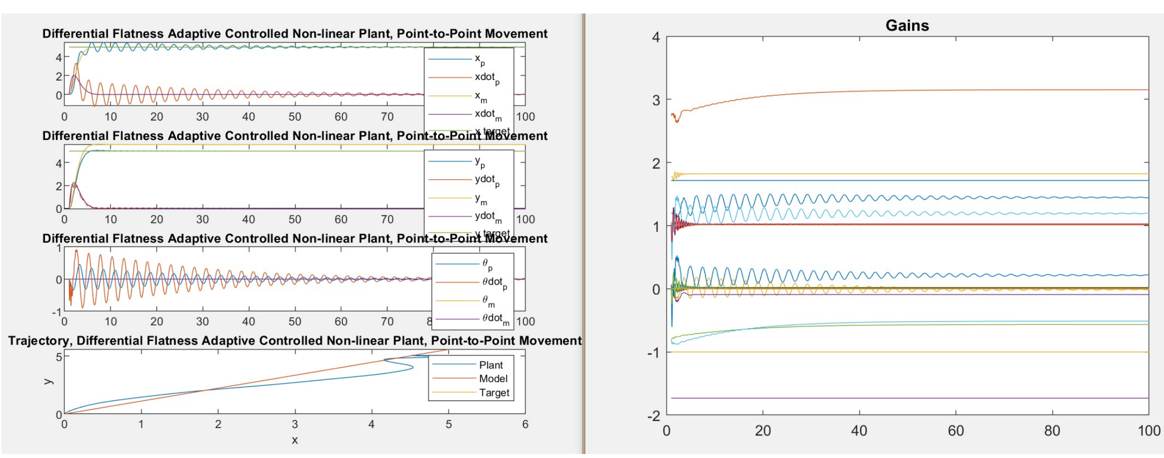
Adaptive Contoller with physical parameters changed by $\pm 20\%$

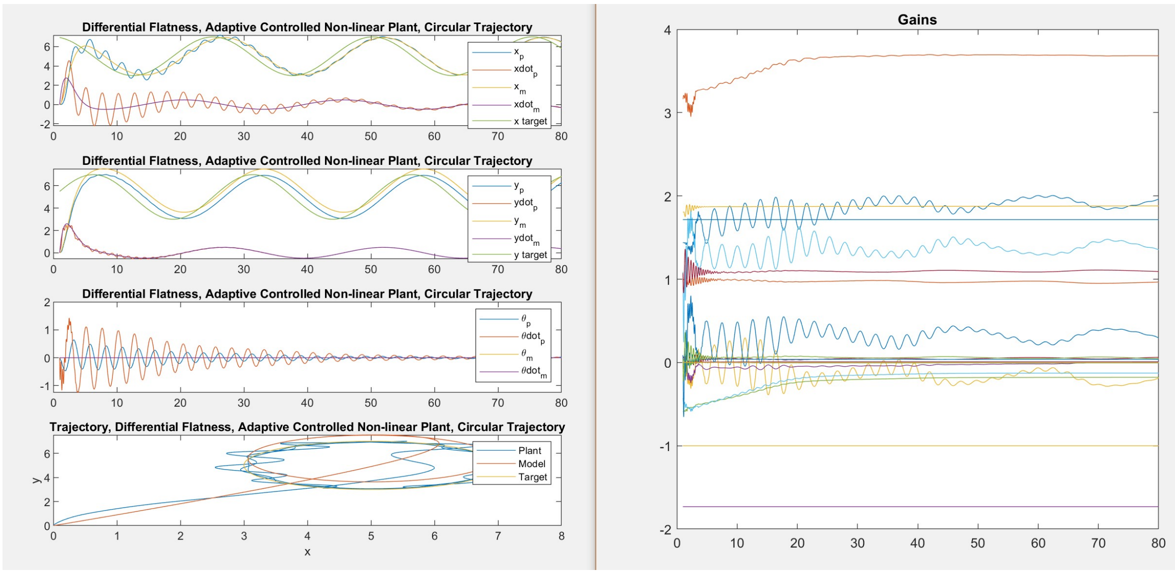




Again, physical parameters changed by $\pm 20\%$, except with the adaptive contoller, it reached the target by following the model. It took more time for the gains to adjust for the difference in parameters. Although the fixed-gain controller is smoother, al least the adaptive controller goes to the correct destination.

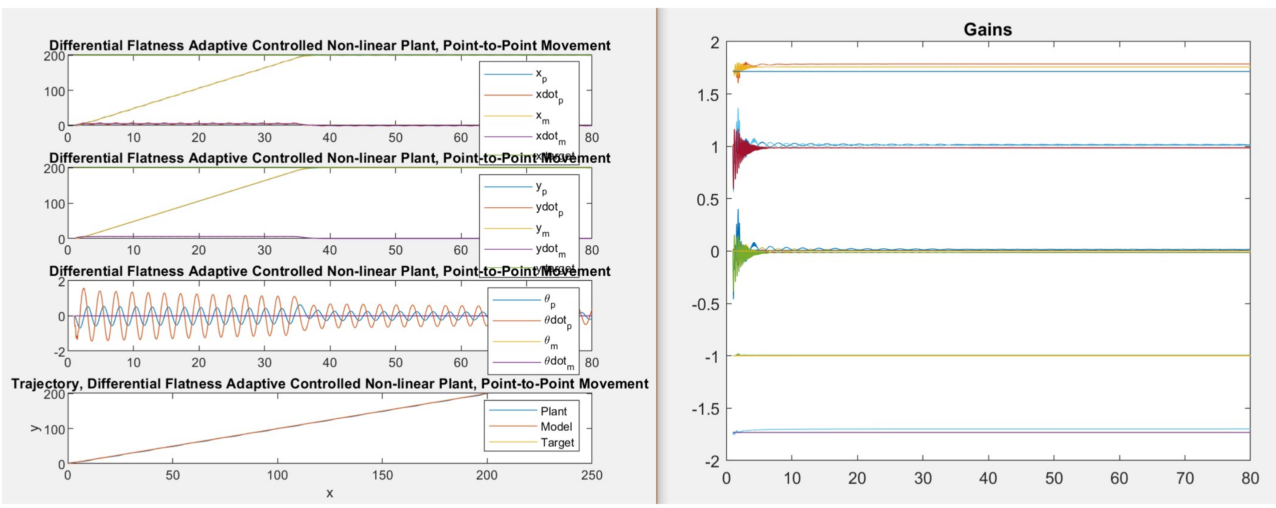
Adaptive Contoller with physical parameters changed by $\pm 20\%$ starting with Gains after Adaptation





It seems that even though it learnt afte adaptation, the non-linearities force a change in the ideal gains, causing it to need to learn the gains again. Although, it does perform much better.

Specs and Conclusion



Simulation with 10-20% variation of physical constants, with reference command far away.

With differential flatness with a non-linear controller and reference command shaping, our birotor helicopter can go to any desired location, with a speed of 6m/s, and a settling time of approximately 15 seconds in a transient response with no overshoot unlike step 1.

Settling time is large to avoid overexherting the thrusters. The non-linear controller took longer to adapt (30 seconds) thant the linear controller with our setup, but the non-linear controller guarantees function far from the equilibrium.

It seems that if the exact pysical parameters are know, adaptive control is more valuable to a linear controller than a non-linear one, because it allows it to adapt to non-lineaer behavior. But the non-linear controller will perform exactly like the model anyway.